Ideals of cubic algebras and an invariant ring of the Weyl algebra

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slides and outline talk available

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<u>Contents:</u>

1. Introduction and main results

- $k = \mathbb{C}$ field of complex numbers
- First Weyl algebra

$$A_1 = k\langle x, y\rangle/(xy - yx - 1)$$

Question: describe right ideals of A_1

- Cannings and Holland (1994), Wilson (1998)

$$R(A_1) = \{\,\text{right } A_1\text{-ideals}\,\}/_{\cong} \longleftrightarrow \coprod_n C_n$$

where

$$C_n = \{(\mathbb{X}, \mathbb{Y}) \in M_n(k)^2 \mid \mathrm{rk}(\mathbb{Y}\mathbb{X} - \mathbb{X}\mathbb{Y} - \mathbb{I}) \leq 1\}/\mathrm{Gl}_n(k)$$

is the n-th Calogero-Moser space

Le Bruyn (1995) proposed an alternative method

- Beilinson (1979)

$$\mathrm{D}^b(\mathrm{coh}(\mathbb{P}^2)) \underset{-\overset{\mathbf{L}}{\otimes}_\Delta \mathcal{E}}{\overset{\mathrm{RHom}_{\mathbb{P}^2}(\mathcal{E},-)}{\rightleftarrows}} \mathrm{D}^b(\mathrm{mod}(\Delta))$$

  where $\mathcal{E} = \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}$ and Δ is the quiver

$$-2 \;\overset{X_{-2}}{\underset{Z_{-2}}{\overset{\xrightarrow{Y_{-2}}}{\Rightarrow}}}\; -1 \;\overset{X_{-1}}{\underset{Z_{-1}}{\overset{\xrightarrow{Y_{-1}}}{\Rightarrow}}}\; 0$$

  with relations reflecting the relations in $k[x, y, z]$

$$\begin{cases} Y_{-1}X_{-2} = X_{-1}Y_{-2} \\ Z_{-1}Y_{-2} = Y_{-1}Z_{-2} \\ X_{-1}Z_{-2} = Z_{-1}X_{-2} \end{cases}$$

- Hulek, Barth (1977-1980): (stable) vector bundles on $\mathbb{P}^2$ are determined by certain (stable) representations of the quiver Δ

- The picture survives when we replace
  - $k[x,y,z]$ by homogenized Weyl algebra

    $H = k\langle x,y,z\rangle \Big/ (zx - xz, zy - yz, yx - xy - z^2)$

    a noncommutative analogue of $k[x,y,z]$
  - $\mathbb{P}^2$ by $\operatorname{Proj} H$ (sense of Artin and Zhang)
  - relations of Δ by the ones induced by H

- Equality $A_1 = H[z^{-1}]_0$ induces bijection between sets

  $$R(A_1) = \{\,\text{right } A_1\text{-ideals}\,\}/_{\cong}$$

  and

  $R(H) = \{\,\text{reflexive graded right } H\text{-ideals}\,\}/_{\cong,\text{sh}}$

  They correspond to "line bundles" on $\mathbb{P}^2_{\mathrm{q}}$

  By derived equivalence line bundles are determined by certain stable representations of the quiver Δ

- Berest and Wilson (2002) used these ideas to reprove

  Theorem A.
  Aut A_1 *has a natural action on* $R(A_1)$
  - *orbits indexed by* $\mathbb{N}$,
  - *the* n*-th orbit is in bijection with* n*-th Calogero-Moser space*

  $$C_n = \{(\mathbb{X}, \mathbb{Y}) \in M_n(k)^2 \,|\, \mathrm{rk}(\mathbb{Y}\mathbb{X} - \mathbb{X}\mathbb{Y} - \mathbb{I}) \leq 1\}/\mathrm{Gl}_n(k)$$

  smooth connected affine variety of dimension $2n$.

There are many more k-algebras inducing a $\mathbb{P}^2_{\mathsf{q}}$.
Interesting class: by Artin and Schelter (1986)

- *Artin-Schelter algebra of dimension* 3 is

  (i) graded k-algebra $A = k \oplus A_1 \oplus A_2 \oplus \ldots$ global dimension 3

  (ii) A has polynomial growth

  (iii) A is Gorenstein, i.e. for some $l \in \mathbb{Z}$

$$\underline{\mathrm{Ext}}^i_A(k_A, A) \cong \begin{cases} {}_A k(l) & \text{if } i = 3, \\ 0 & \text{otherwise.} \end{cases}$$

- Classified by Artin, Tate and Van den bergh (1990) and Stephenson (1994).

- They are all noetherian domains GK-dim 3 have all expected nice homological properties

- Assume A generated in degree one.
  Two possibilities:

  - A is *quadratic*

    $0 \to A(-3) \to A(-2)^3 \to A(-1)^3 \to A \to k_A \to 0$

    $$h_A(t) = \sum_n \dim_k A_n t^n = \frac{1}{(1-t)^3}$$

  - A is *cubic*

    $0 \to A(-4) \to A(-3)^2 \to A(-1)^2 \to A \to k_A \to 0$

    $$h_A(t) = \sum_n \dim_k A_n t^n = \frac{1}{(1-t)^2(1-t^2)}$$

- Generic class: called type A algebras

  $$\text{quadratic:} \begin{cases} ayz + bzy + cx^2 = 0 \\ azx + bxz + cy^2 = 0 \\ axy + byx + cz^2 = 0 \end{cases}$$

  $$\text{cubic:} \begin{cases} ay^2x + byxy + axy^2 + cx^3 = 0 \\ ax^2y + bxyx + ayx^2 + cy^3 = 0 \end{cases}$$

  where $a, b, c \in k$ generic.

- A is determined by a triple (E, σ, j) where either

  - A is *linear*:
    $j : E \cong \mathbb{P}^2$ if A is quadratic
    $j : E \cong \mathbb{P}^1 \times \mathbb{P}^1$ if A is cubic

  - A is *elliptic*:
    $j : E \hookrightarrow \mathbb{P}^2$, E divisor degree 3
    if A is quadratic
    $j : E \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$, E divisor bidegree $(2, 2)$
    if A is cubic

  Generic case: A is type A and E is smooth elliptic curve (called *generic type A*)

- Define the set

  $R(A) = \{$ reflexive graded right A-ideals$\}/_{\cong,\text{sh}}$

- Van den Bergh and De Naeghel (2002)

  Theorem B.
  Let A *be elliptic quadratic and* $o(\sigma) = \infty$*.*
  Then the set $R(A)$ *is in bijection with* $\coprod_{n\in\mathbb{N}} D_n$ *where* D_n *is smooth locally closed variety dimension* $2n$*.*
  If A *is of generic type A then* D_n *is affine.*

- Similar result by Nevins and Stafford (2002).

- Aim of the talk:
  analogue of Theorem B for cubic A.

$$N := \{(n_e, n_o) \in \mathbb{N}^2 \mid n_e - (n_e - n_o)^2 \geq 0\}$$

  Theorem 1.
  Let A *be elliptic cubic and* $o(\sigma) = \infty$*.*
  Then the set $R(A)$ *is in bijection with* $\coprod_{(n_e,n_o)\in N} D_{(n_e,n_o)}$ *where* $D_{(n_e,n_o)}$ *is smooth locally closed variety of dimension* $2(n_e - (n_e - n_o)^2)$*.*
  If A *is of generic type A then* $D_{(n_e,n_o)}$ *is affine.*

- Application: enveloping algebra

$$H_c = k\langle x,y,z\rangle/(yz-zy, xz-zx, xy-yx-z)$$
$$= k\langle x,y\rangle/([y,[y,x]],[x,[x,y]])$$

Let $\varphi \in \text{Aut}(A_1)$, $\varphi(x) = -x$, $\varphi(y) = -y$.

Equality $A_1^{\langle\varphi\rangle} = H_c[z^{-1}]_0$ induces bijection between sets

$$R(A_1^{\langle\varphi\rangle}) = \{\,\text{right } A_1^{\langle\varphi\rangle}\text{-ideals}\,\}/_{\cong}$$

and

$$R(H_c) = \{\,\text{reflexive graded right } H_c\text{-ideals}\,\}/_{\cong,\text{sh}}$$

By Theorem 1 and further investigation

Theorem 2. *The set $R(A_1^{\langle\varphi\rangle})$ is in bijection with $\coprod_{(n_e,n_o)\in N} D_{(n_e,n_o)}$ where*

$$D_{(n_e,n_o)} = \{(\mathbb{X},\mathbb{Y},\mathbb{X}',\mathbb{Y}') \in M_{n_e\times n_o}(k)^2 \times M_{n_o\times n_e}(k)^2 \mid \mathbb{Y}'\mathbb{X}-\mathbb{X}'\mathbb{Y} = \mathbb{I} \text{ and } \text{rank}(\mathbb{Y}\mathbb{X}'-\mathbb{X}\mathbb{Y}'-\mathbb{I}) \le 1\}\,/\text{Gl}_{n_e}(k)\times\text{Gl}_{n_o}(k)$$

are smooth affine varieties of dimension $2(n_e-(n_e-n_o)^2)$.

2. Noncommutative quadrics

- A cubic Artin-Schelter algebra
  graded k algebra $A = k \oplus A_1 \oplus A_2 \oplus \ldots$
  Hilbert series $1 + 2t + 4t^2 + 6t^3 + 9t^4 + \ldots$

- graded right A-module $M = \oplus_{i\in\mathbb{Z}} M_i$ then

$$\begin{array}{c} \text{degree } 0 \\ \downarrow \end{array}$$

$$M = \cdots \oplus M_{-1} \oplus M_0 \oplus M_1 \oplus \ldots$$

$$M(1) := \cdots \oplus M_0 \ \oplus M_1 \oplus M_2 \oplus \ldots$$

- \- GrMod(A) graded right A-modules

  \- Tors(A) direct limits of fdim modules

  \- GrMod(A) $\xrightarrow{\pi}$ GrMod(A)/ Tors(A) = Tails(A)
  $M \mapsto \mathcal{M}$
  $A \mapsto \mathcal{O}$

  \- $M \mapsto M(1)$ induces sh : $\mathcal{M} \mapsto \mathcal{M}(1)$

  \- grmod(A), tors(A), tails(A) noeth. obj.

  \- $X :=$ Proj(A) := (tails(A), $\mathcal{O}$, sh)

  \- Qcoh(X) := Tails(A), coh(X) := tails(A)

A is determined by triple (E, σ, j)

- $P = \oplus_n P_n \in \text{grmod}(A)$ is a *point module* if

$$A \twoheadrightarrow P \text{ and } h_P(t) = \sum_n \dim_k P_n t^n = \frac{1}{1-t}$$

Choosing a k-basis $e_0, e_1, \ldots$ in $P_0, P_1, \ldots$

$$\begin{cases} e_0 \cdot x = \alpha_0 e_1 \\ e_0 \cdot y = \beta_0 e_1 \end{cases}, \begin{cases} e_1 \cdot x = \alpha_1 e_2 \\ e_1 \cdot y = \beta_1 e_2 \end{cases}, \begin{cases} e_2 \cdot x = \alpha_2 e_3 \\ e_2 \cdot y = \beta_2 e_3 \end{cases}, \ldots$$

for some $\alpha_i, \beta_i \in k$.

If r is relation in A then $e_0 \cdot r = 0$.

Leads to equation in $(\alpha_0, \beta_0), (\alpha_1, \beta_1)$

E is the zero locus in $\mathbb{P}^1 \times \mathbb{P}^1$ of equation.

- Example: if A is type A then E is

$$(c^2 - b^2)\alpha_0\beta_0\alpha_1\beta_1 + a\alpha_0^2(c\alpha_1^2 - b\beta_1^2) + a\beta_0^2(c\beta_1^2 - b\alpha_1^2) = 0$$

- Example: if $A = H_c$ enveloping algebra then E is

$$(\alpha_0\beta_1 - \beta_0\alpha_1)^2 = 0$$

the double diagonal $2D$ on $\mathbb{P}^1 \times \mathbb{P}^1$

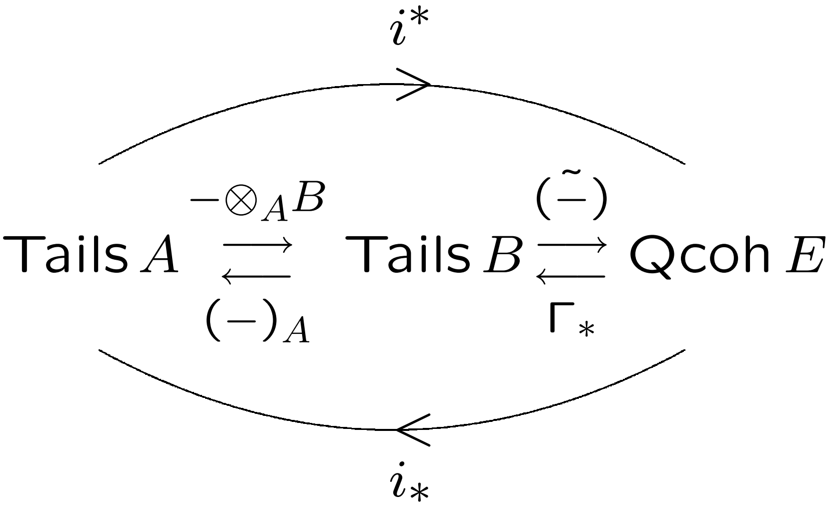
- Point module $P \mapsto p$ closed point in E

$$P(1)_{\geq 0} \mapsto \sigma(p)$$

Artin, Tate and Van den Bergh:

$$A \longrightarrow (E, \sigma, j) \longrightarrow B = B(E, \sigma, j)$$

- A linear: $B \cong A$
  A elliptic: $B \cong A/gA$ with $g \in A_4$ normal
- Tails B and Qcoh E are equivalent



i^* is right exact (*restriction functor*)
i_* is exact

3. From reflexive ideals to line bundles

- A cubic Artin-Schelter algebra, $X = \operatorname{Proj} A$

- For $I \in \operatorname{grmod}(A)$ graded right ideal

$$m \mapsto \dim_k A_m - \dim_k I_m$$

is linear in m.

- For $J \in \operatorname{grmod}(A)$ of rank one, $\exists!\, d \in \mathbb{Z}$ s.t.

$$\dim_k A_m - \dim_k J(d)_m = \begin{cases} n_e & \text{for } m \gg 0 \text{ even} \\ n_o & \text{for } m \gg 0 \text{ odd} \end{cases}$$

for some integers n_e, n_o. We say

- $J(d)$ is *normalized*

- (n_e, n_o) are the *invariants* of J

- For $n_e, n_o \in \mathbb{Z}$ define full subcategory of $\operatorname{grmod}(A)$

$$R_{(n_e,n_o)}(A) = \{J \in \operatorname{grmod}(A) \mid J \text{ rank one, reflexive, normalized with invariants } (n_e, n_o)\}$$

- Apply exact quotient functor

$$\pi : \mathsf{grmod}(A) \to \mathsf{coh}(X)$$
$$I \mapsto \mathcal{I}$$

Let $\mathcal{R}_{(n_e,n_o)}(X)$ be image of $R_{(n_e,n_o)}(A)$. Objects in $\mathcal{R}_{(n_e,n_o)}(X)$ are called *normalized line bundles.*

- $R_{(n_e,n_o)}(A)$ and $\mathcal{R}_{(n_e,n_o)}(X)$ are groupoids

- Natural bijection between set

$R(A) = \{\text{ reflexive graded right } A\text{-ideals}\}/_{\cong,\mathsf{sh}}$

and isoclasses of

$$\coprod_{(n_e,n_o)\in\mathbb{Z}^2} R_{(n_e,n_o)}(A)$$

and isoclasses of

$$\coprod_{(n_e,n_o)\in\mathbb{Z}^2} \mathcal{R}_{(n_e,n_o)}(X)$$

- For $\mathcal{I} \in \mathcal{R}_{(n_e,n_o)}(X)$ its *cohomology groups*

$$H^i(X,\mathcal{I}) := \text{Ext}^i_X(\mathcal{O},\mathcal{I})$$

are partially computed (for $\mathcal{I} \ncong \mathcal{O}$)

l	...	-5	-4	-3	-2	-1	0	1	...
$\dim_k H^0(X,\mathcal{I}(l))$	...	0	0	0	0	0	0	$*$	...
$\dim_k H^1(X,\mathcal{I}(l))$	...	$*$	n_e-1	n_o	n_e	n_o	n_e-1	$*$	...
$\dim_k H^2(X,\mathcal{I}(l))$	...	$*$	0	0	0	0	0	0	...

Thus $n_e > 0$, $n_o \geq 0$ and easy to prove

$$\mathcal{R}_{(0,0)} = \{\mathcal{O}\}$$

- By standard computation on the Euler form

$$\dim_k \text{Ext}^1_X(\mathcal{I},\mathcal{I}) = 2(n_e - (n_e - n_o)^2)$$

hence for any integers n_e, n_o

$$\mathcal{R}_{(n_e,n_o)}(X) \neq \emptyset \Rightarrow (n_e,n_o) \in N$$

Converse also true! See below.

4. From line bundles to quiver representations

- A cubic Artin-Schelter algebra, $X = \operatorname{Proj} A$
- Due to a Theorem of Bondal (1990)

$$\mathsf{D}^b(\operatorname{coh} X) \underset{- \overset{\mathbf{L}}{\otimes}_\Gamma \mathcal{E}}{\overset{\mathsf{RHom}_X(\mathcal{E},-)}{\rightleftarrows}} \mathsf{D}^b(\operatorname{mod} \Gamma)$$

where $\mathcal{E} = \mathcal{O}(3) \oplus \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}$
and Γ is the quiver

$$-3 \overset{X_{-3}}{\underset{Y_{-3}}{\rightrightarrows}} -2 \overset{X_{-2}}{\underset{Y_{-2}}{\rightrightarrows}} -1 \overset{X_{-1}}{\underset{Y_{-1}}{\rightrightarrows}} 0$$

with relations reflecting the relations in A.

- Try to understand image of $\mathcal{R}_{(n_e,n_o)}(X)$

Assume $(n_e, n_o) \neq (0,0)$. Fix $\mathcal{I} \in \mathcal{R}_{(n_e,n_o)}(X)$.

- Consider $\mathcal{I}$ as complex of degree zero

- By previous its image is in degree one

$$\mathrm{RHom}_X(\mathcal{E},\mathcal{I}) = M[-1]$$

where $M = \mathrm{Ext}^1_X(\mathcal{E},\mathcal{I})$

- M is given by a representation of Γ

$$H^1(X,\mathcal{I}(-3)) \overset{X}{\underset{Y}{\rightrightarrows}} H^1(X,\mathcal{I}(-2)) \overset{X'}{\underset{Y'}{\rightrightarrows}} H^1(X,\mathcal{I}(-1)) \overset{X''}{\underset{Y''}{\rightrightarrows}} H^1(X,\mathcal{I})$$

$\underline{\mathrm{dim}} M = (n_o, n_e, n_o, n_e - 1)$ and relations.

For example, if A is type A then

$$\begin{pmatrix} X'' & Y'' \end{pmatrix} \cdot \begin{pmatrix} aY'Y + cX'X & bX'Y + aY'X \\ bY'X + aX'Y & aX'X + cY'Y \end{pmatrix} = 0$$

How is "$\mathcal{I}$ is line bundle" translated?

- Let P be point module, $\mathcal{P} = \pi P$.
  Cohomology groups given by

l	$\ldots$	-5	-4	-3	-2	-1	0	1	$\ldots$
$\dim_k H^0(X, \mathcal{P}(l))$	$\ldots$	1	1	1	1	1	1	1	$\ldots$
$\dim_k H^1(X, \mathcal{P}(l))$	$\ldots$	0	0	0	0	0	0	0	$\ldots$
$\dim_k H^2(X, \mathcal{P}(l))$	$\ldots$	0	0	0	0	0	0	0	$\ldots$

  $\mathcal{P}$ determines the representation of Γ

$$k \underset{\beta_{-3}}{\overset{\alpha_{-3}}{\rightrightarrows}} k \underset{\beta_{-2}}{\overset{\alpha_{-2}}{\rightrightarrows}} k \underset{\beta_{-1}}{\overset{\alpha_{-1}}{\rightrightarrows}} k$$

  where $p = ((\alpha_0, \beta_0), (\alpha_1, \beta_1)) \in E$
  $\sigma^i p = ((\alpha_i, \beta_i), (\alpha_{i+1}, \beta_{i+1}))$

- $\mathcal{I}$ is line bundle means $\mathrm{Ext}^1_X(\mathcal{P}, \mathcal{I}) = 0$

$$\begin{aligned} 0 = \mathrm{Ext}^1_X(\mathcal{P}, \mathcal{I}) &= H^0(\mathrm{RHom}_X(\mathcal{P}, \mathcal{I}[1])) \\ &\cong H^0(\mathrm{RHom}_\Gamma(p, M)) \\ &= \mathrm{Hom}_\Gamma(p, M) \end{aligned}$$

- Trivially,

$$\begin{aligned}\mathrm{Hom}_{\Delta}(M,p) &= H^0(\mathrm{RHom}_{\Gamma}(M,p)) \\ &= H^0(\mathrm{RHom}_X(\mathcal{I}[1],\mathcal{P})) = 0\end{aligned}$$

- These properties characterize M.

Theorem.
Let A be elliptic cubic and $o(\sigma)=\infty$.
Then equivalence

$$\mathcal{R}_{(n_e,n_o)}(X) \underset{\mathrm{Tor}_1^{\Gamma}(-,\mathcal{E})}{\overset{\mathrm{Ext}_X^1(\mathcal{E},-)}{\rightleftarrows}} \mathcal{C}_{(n_e,n_o)}(\Gamma)$$

where

$$\begin{aligned}\mathcal{C}_{(n_e,n_o)}(\Gamma) = \{M \in \mathrm{mod}(\Gamma) \mid \underline{\mathrm{dim}}M = (n_o,n_e,n_o,n_e-1) \\ \mathrm{Hom}_{\Gamma}(M,p) = \mathrm{Hom}_{\Gamma}(p,M) = 0\ \forall p \in E\}\end{aligned}$$

- Problem: $\mathcal{C}_{(n_e,n_o)}(\Gamma)$ not easy to handle

Pick up another idea of Le Bruyn.

$\mathcal{I} \in \mathcal{R}_{(n_e,n_o)}(X)$ is determined by both

$$H^1(X,\mathcal{I}(-3)) \overset{X}{\underset{Y}{\rightrightarrows}} H^1(X,\mathcal{I}(-2)) \overset{X'}{\underset{Y'}{\rightrightarrows}} H^1(X,\mathcal{I}(-1)) \overset{X''}{\underset{Y''}{\rightrightarrows}} H^1(X,\mathcal{I})$$

$$H^1(X,\mathcal{I}(-4)) \overset{'X}{\underset{'Y}{\rightrightarrows}} H^1(X,\mathcal{I}(-3)) \overset{X}{\underset{Y}{\rightrightarrows}} H^1(X,\mathcal{I}(-2)) \overset{X'}{\underset{Y'}{\rightrightarrows}} H^1(X,\mathcal{I}(-1))$$

So $\mathcal{I}$ is actually determined by repr. M^0

$$H^1(X,\mathcal{I}(-3)) \overset{X}{\underset{Y}{\rightrightarrows}} H^1(X,\mathcal{I}(-2)) \overset{X'}{\underset{Y'}{\rightrightarrows}} H^1(X,\mathcal{I}(-1))$$

of the full subquiver Γ^0 of Γ

$$-3 \overset{X_{-3}}{\underset{Y_{-3}}{\rightrightarrows}} -2 \overset{X_{-2}}{\underset{Y_{-2}}{\rightrightarrows}} -1$$

Characterizing properties of M^0?

- As M^0 is the restriction of M: certain rank condition involving X, Y, X', Y'

  For example, if A is type A then

$$\begin{pmatrix} X'' & Y'' \end{pmatrix} \cdot \begin{pmatrix} aY'Y + cX'X & bX'Y + aY'X \\ bY'X + aX'Y & aX'X + cY'Y \end{pmatrix} = 0$$

  yields

$$\operatorname{rank} \begin{pmatrix} aY'Y + cX'X & bX'Y + aY'X \\ bY'X + aX'Y & aX'X + cY'Y \end{pmatrix}$$
$$\leq \dim \ker \begin{pmatrix} X'' & Y'' \end{pmatrix} = 2n_o - (n_e - 1)$$

- M^0 is θ-stable for $\theta = (-1, 0, 1)$.
  Indeed:

$$\forall \mathcal{I} \in \mathcal{R}_{(n_e,n_o)}(X) : \exists\, v \in A_2 : \mathrm{Hom}_X(\mathcal{I}, \pi(A/vA)) = 0$$
$$\Rightarrow \mathrm{Hom}_{\Gamma^0}(M^0, Q^0) = 0$$
$$\Rightarrow M^0 \perp Q^0$$

  for some representation $Q^0 \in \mathrm{mod}(\Gamma^0)$.

- These properties characterize M^0 for $(n_e, n_o) \neq (1,1)$

Theorem.
Let A be elliptic cubic and $o(\sigma) = \infty$. Then equivalence

$$\mathcal{C}_{(n_e,n_o)}(\Gamma) \underset{\text{Ind}}{\overset{\text{Res}}{\rightleftarrows}} \mathcal{D}_{(n_e,n_o)}(\Gamma^0)$$

where

$$\mathcal{D}_{(n_e,n_o)}(\Gamma^0) = \{F \in \text{mod}(\Gamma^0) \mid \underline{\text{dim}} F = (n_o, n_e, n_o), \\ F \textit{ is } \theta\textit{-stable}, \ \dim_k(\text{Ind}\, F)_0 \geq n_e - 1\}$$

- Put $\alpha = (n_o, n_e, n_o)$ and

$$D_{(n_e,n_o)} := \{F \in \text{Rep}_\alpha(\Gamma^0) \mid F \in \mathcal{D}_{(n_e,n_o)}(\Gamma^0)\} /\!\!/ \, \text{Gl}_\alpha(k)$$

Locally closed follows.
Smooth of dimension $2(n_e - (n_e - n_o)^2)$ is proved.

5. Generic type A

Assume A is generic type A.

$\coprod_{(n_e,n_o)} \mathcal{R}_{(n_e,n_o)}(X)$ equivalent to

$$\{\mathcal{M} \in \mathsf{coh}(X) \mid i^*\mathcal{M} \text{ is line bundle on } E \text{ deg. zero}\}$$

By picking a suitable line bundle $\mathcal{V}$ on E

$$\begin{aligned}\forall \mathcal{I} \in \mathcal{R}_{(n_e,n_o)}(X) : \mathsf{RHom}_E(i^*\mathcal{I}, \mathcal{V}) = 0 \\ \Rightarrow \mathsf{RHom}_X(\mathcal{I}, i_*\mathcal{V}) = 0 \\ \Rightarrow \mathsf{RHom}_{\Gamma^0}(M^0, V^0) = 0 \\ \Rightarrow M^0 \perp V^0\end{aligned}$$

for some representation $V^0 \in \mathsf{mod}(\Gamma^0)$.

Leads to

$$\begin{aligned}\mathcal{D}_{(n_e,n_o)}(\Gamma^0) = \{F \in \mathsf{mod}(\Gamma^0) \mid \underline{\mathsf{dim}} F = (n_o, n_e, n_o), \\ F \perp V^0, \mathsf{dim}_k(\mathsf{Ind}\, F)_0 \geq n_e - 1\}\end{aligned}$$

As $\{F \in \mathsf{Rep}_\alpha(\Gamma^0) \mid F \perp V^0\}$ is affine we deduce $D_{(n_e,n_o)}$ is also affine.

6. The enveloping algebra

Assume $A = H_c$ is the enveloping algebra

$$\begin{aligned} H_c &= k\langle x, y, z\rangle/(yz - zy, xz - zx, xy - yx - z) \\ &= k\langle x, y\rangle/([y,[y,x]],[x,[x,y]]) \end{aligned}$$

Work with $E_{\mathrm{red}} = D$, the diagonal on $\mathbb{P}^1 \times \mathbb{P}^1$.
$\coprod_{(n_e,n_o)} \mathcal{R}_{(n_e,n_o)}(X)$ equivalent to

$$\begin{aligned} &\{\mathcal{M} \in \mathrm{coh}(X) \mid i^*\mathcal{M} \text{ line bundle on } D \text{ deg. zero}\} \\ =&\{\mathcal{M} \in \mathrm{coh}(X) \mid i^*\mathcal{M} \cong \mathcal{O}_D\} \end{aligned}$$

Translates into

$$\begin{aligned} \forall \mathcal{I} \in \mathcal{R}_{(n_e,n_o)}(X) : \mathrm{Hom}_X(\mathcal{I}, \pi(A/zA)) = 0 \\ \Rightarrow \mathrm{Hom}_{\Gamma^0}(M^0, Q^0) = 0 \\ \Rightarrow M^0 \perp Q^0 \end{aligned}$$

Leads to

$$\begin{aligned} \mathcal{D}_{(n_e,n_o)}(\Gamma^0) = \{F \in \mathrm{mod}(\Gamma^0) \mid \underline{\mathrm{dim}} F = (n_o, n_e, n_o), \\ F \perp Q^0, \mathrm{dim}_k(\mathrm{Ind}\, F)_0 \geq n_e - 1\} \end{aligned}$$

Thus $D_{(n_e,n_o)}$ is affine.

We now simplify this expression.
Let $F \in \mathrm{Rep}_\alpha(\Gamma^0)$ i.e.

$$k^{n_o} \underset{Y}{\overset{X}{\rightrightarrows}} k^{n_e} \underset{Y'}{\overset{X'}{\rightrightarrows}} k^{n_o}$$

- $F \perp Q^0$ means $Y'X - X'Y$ is an isomorphism
  Defining

  $$\begin{cases} \mathbb{X} = X \\ \mathbb{Y} = Y \end{cases} \text{ and } \begin{cases} \mathbb{X}' = (Y'X - X'Y)^{-1}X' \\ \mathbb{Y}' = (Y'X - X'Y)^{-1}Y' \end{cases}$$

  this means $\mathbb{Y}\mathbb{X}' - \mathbb{X}'\mathbb{Y}' = \mathbb{I}$

- $\dim_k(\mathrm{Ind}\, F)_0 \geq n_e - 1$ means

$$\mathrm{rank} \begin{pmatrix} Y'Y & X'Y - 2Y'X \\ Y'X - 2X'Y & X'X \end{pmatrix} \leq 2n_o - (n_e - 1)$$

which means $\mathrm{rank}(\mathbb{Y}\mathbb{X}' - \mathbb{X}\mathbb{Y}' - \mathbb{I}) \leq 1$

7. Hilbert series of ideals

A is three-dimensional Artin-Schelter algebra.

Question.
If $I = \oplus_n I_n \in \mathrm{grmod}(A)$ is a graded right ideal, how does $h_I(t) = \sum_n \dim_k I_n t^n$ look like?

- Macaulay (1927): for $A = k[x, y, z]$
  Restrict to $\mathrm{pd}\, I \leq 1$.

  $$m \mapsto \dim_k A_m - \dim_k I_m$$

  is linear in m, i.e. $\exists!\, d \in \mathbb{Z}$ s.t.

  $$\dim_k A_m - \dim_k I(d)_m = n \text{ for } m \gg 0$$

  for some integer n.
  In terms of formal power series

  $$\begin{aligned} h_{I(d)}(t) &= h_{k[x,y,z]}(t) - \frac{s(t)}{1-t} \\ &= \frac{1}{(1-t)^3} - \frac{s(t)}{1-t} \end{aligned}$$

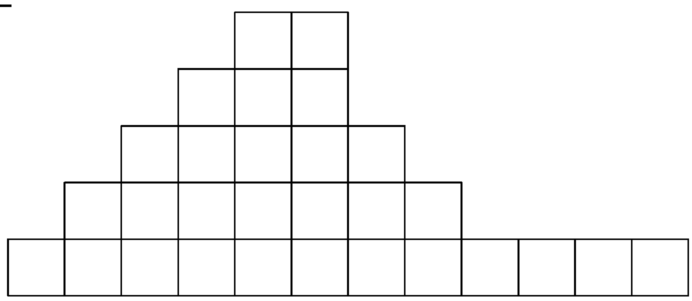
  for some $s(t) \in \mathbb{Z}[t, t^{-1}]$ with $s(1) = n$.

Turns out $s(t)$ is a *Castelnuovo polynomial*

$$s(t) = 1 + 2t + 3t^2 + \cdots + ut^{u-1} + s_u t^u + \cdots + s_v t^v$$
$$u \geq s_u \geq \ldots \geq s_v \geq 0$$

for some integers $u, v \geq 0$.
$s(1) = n \geq 0$.

Visualized in form of a stair
Example:



1 2 3 4 5 5 3 2 1 1 1 1

Answer (commutative case).
$h(t) \in \mathbb{Z}((t))$ is of the form $h_{I(d)}(t)$ for some graded ideal I with pd $I \leq 1$ *if and only if*

$$h(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t}$$

for some Castelnuovo polynomial $s(t)$.

- Van den Bergh and De Naeghel (2004):

  Answer (quadratic case).
  Let A be quadratic Artin-Schelter algebra. $h(t) \in \mathbb{Z}((t))$ is of the form $h_{I(d)}(t)$ for some graded right ideal I with $\operatorname{pd} I \leq 1$ iff

$$h(t) = \frac{1}{(1-t)^3} - \frac{s(t)}{1-t}$$

  for some Castelnuovo polynomial $s(t)$.

  If in addition A is elliptic and $o(\sigma) = \infty$ then I may be chosen reflexive.

- What is the answer for cubic A?

  As indicated above: $\exists!\, d \in \mathbb{Z}$ s.t.

$$\dim_k A_m - \dim_k I(d)_m = \begin{cases} n_e & \text{for } m \gg 0 \text{ even} \\ n_o & \text{for } m \gg 0 \text{ odd} \end{cases}$$

  for some integers $n_e, n_o \geq 0$.

  In terms of formal power series

$$h_I(t) = \frac{1}{(1-t)^2(1-t^2)} - \frac{s(t)}{1-t^2}$$

  for some $s(t) = \sum_i s_i t^i \in \mathbb{Z}[t, t^{-1}]$
  with $\sum_i s_{2i} = n_e$ and $\sum_i s_{2i+1} = n_o$.

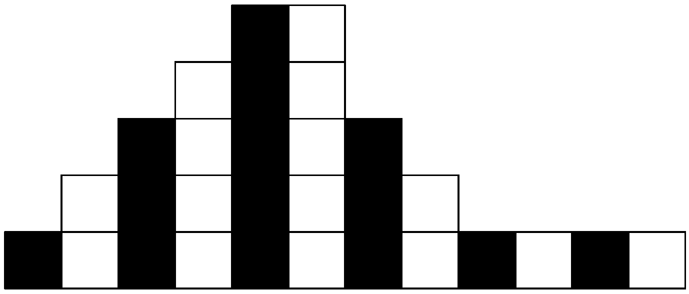
  Answer (cubic case).
  Let A be cubic Artin-Schelter algebra. $h(t) \in \mathbb{Z}((t))$ is of the form $h_{I(d)}(t)$ for some graded right ideal I with $\operatorname{pd} I \leq 1$ iff

$$h(t) = \frac{1}{(1-t)^2(1-t^2)} - \frac{s(t)}{1-t^2}$$

  for some Castelnuovo polynomial $s(t)$.

  If in addition A is elliptic and $o(\sigma) = \infty$ then I may be chosen reflexive.

Example:



1 2 3 4 5 5 3 2 1 1 1 1

$\sum_i s_{2i} = 14$ *even weight*
$\sum_i s_{2i+1} = 15$ *odd weight*.

As a consequence, for all integers n_e, n_o

$$\mathcal{R}_{(n_e,n_o)}(X) \neq \emptyset \Leftrightarrow \ \exists \text{ Castelnuovo polynomial } s(t) \text{ with even weight } n_e \text{ and odd weight } n_o$$

But recall

$$\mathcal{R}_{(n_e,n_o)}(X) \neq \emptyset \Rightarrow (n_e, n_o) \in N$$

where

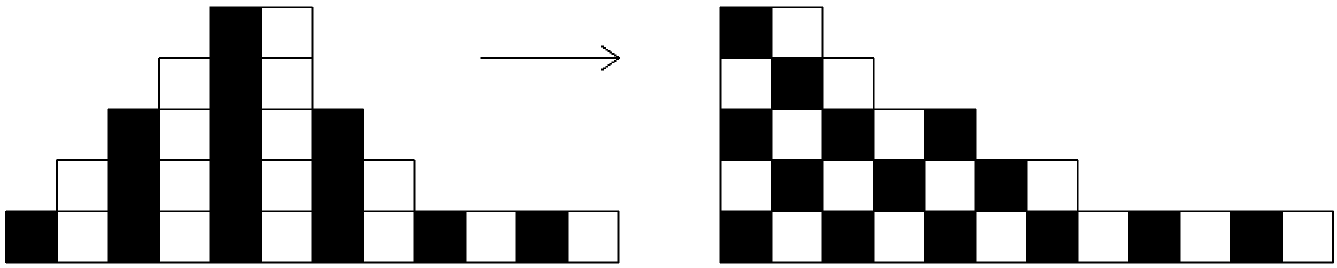
$$N = \{(n_e, n_o) \in \mathbb{N}^2 \mid n_e - (n_e - n_o)^2 \geq 0\}$$

For $(n_e, n_o) \in N$ easy to construct $s(t)$, hence

$$\mathcal{R}_{(n_e,n_o)}(X) \neq \emptyset \Leftrightarrow (n_e, n_o) \in N$$

We end with combinatorical by-product.

Shift rows of $s(t)$ to left



For any partition λ, put draughts colouring.
Number of ■ is called *even weight of* λ
Number of □ is called *odd weight of* λ

Bijection between

$\{$Castelnuovo polynomials
with even weight n_e and odd weight $n_o\}$

and

$\{$partitions in distinct parts
with even weight n_e and odd weight $n_o\}$

Theorem. *Let $n_e, n_o \in \mathbb{N}$. There is a partition λ in distinct parts with even weight n_e and odd weight n_o iff*

$$n_e - (n_e - n_o)^2 \geq 0$$

Special case of Chung's Conjecture (1951)
Proved by G. de B. Robinson (1951)
In particular, the number of such partitions λ is the number of partitions of $n_e - (n_e - n_o)^2$.

We may extend Theorem to all partitions